SOLUTIONS OF THE EINSTEIN EQUATIONS INVOLVING FUNCTIONS OF ONLY ONE VARIABLE*

BY

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Taking the element of length in the orthogonal form

$$(1) \qquad ds^2 = \lambda_1\, dx_1^2 + \lambda_2\, dx_2^2 + \lambda_3\, dx_3^2 + \lambda_4\, dx_4^2$$

we assume that the potentials λ are functions of a single variable x_1. We shall find the forms of these functions for which the Einstein cosmological equations

$$(2) \qquad R_{ik} - \frac{1}{4} R g_{ik} = 0$$

are satisfied. The results are given in (14), (20), and (17) below. They involve only elementary functions, and the corresponding four-dimensional manifolds can be represented as imbedded in flat space of seven dimensions.

Making use of the formulas for R_{ik} calculated in a previous paper† we find that, of the ten equations (2), the six with unlike subscript vanish identically, while the four with like subscripts become

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\begin{aligned}
(3) \quad & L_{211} + L_{311} + L_{411} + L_{21}^2 + L_{31}^2 + L_{41}^2 - L_{21} L_{31} - L_{31} L_{41} \\
& \qquad\qquad - L_{41} L_{21} - L_{11}\,(L_{21} + L_{31} + L_{41}) = 0, \\[2mm]
& L_{211} - L_{311} - L_{411} + L_{21}^2 - L_{31}^2 - L_{41}^2 + L_{21} L_{31} - L_{31} L_{41} \\
& \qquad\qquad + L_{41} L_{21} - L_{11}\,(L_{21} - L_{31} - L_{41}) = 0, \\[2mm]
& -L_{211} + L_{311} - L_{411} - L_{21}^2 + L_{31}^2 - L_{41}^2 - L_{21} L_{31} + L_{31} L_{41} \\
& \qquad\qquad - L_{41} L_{21} - L_{11}\,(-L_{21} + L_{31} - L_{41}) = 0, \\[2mm]
& -L_{211} + L_{311} + L_{411} - L_{21}^2 - L_{31}^2 + L_{41}^2 - L_{21} L_{31} - L_{31} L_{41} \\
& \qquad\qquad + L_{41} L_{21} - L_{11}\,(-L_{21} - L_{31} + L_{41}) = 0.
\end{aligned}
$$

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Here $L_i = \frac{1}{2} \log \lambda_i$ and the second and third subscripts indicate differentiation with respect to x_1, while the first subscript states merely the name of the function.

The four equations (3) are not independent, their sum being identically zero. They thus reduce to three equations. We may assume without loss of generality that $\lambda_1 = 1$, for by applying the transformation

$$\lambda_1 \, dx_1^2 = dx_1^{*2}$$

to the form (1), it reduces to

$$(4) \qquad\qquad ds^2 = dx_1^{*2} + \lambda_2^* \, dx_2^2 + \lambda_3^* \, dx_3^2 + \lambda_4^* \, dx_4^2,$$

where the λ_i^* are three new functions of the new variable x_1^*. Thus the functions L_i reduce to

$$L_1^* = 0, \qquad L_i^* = \frac{1}{2} \log \lambda_i^* \qquad (i = 2, 3, 4).$$

We then obtain from equations (3) by adding the first and second, first and third, and the first and fourth, respectively

$$L_{211}^* + L_{21}^{*2} - L_{31}^* L_{41}^* = 0,$$

$$(5) \qquad\qquad L_{311}^* + L_{31}^* - L_{41}^* L_{21}^* = 0,$$

$$L_{411}^* + L_{41}^* - L_{21}^* L_{31}^* = 0.$$

We now introduce three new functions

$$(5') \qquad\qquad x = L_{21}^*, \qquad y = L_{31}^*, \qquad z = L_{41}^*,$$

and denote the independent variable x_1^* by t, thus reducing our system (5) to the simple form

$$x' = yz - x^2,$$

$$(6) \qquad\qquad y' = zx - y^2,$$

$$z' = xy - z^2,$$

a system of three ordinary differential equations, primes denoting differentiation with respect to t.

SOLUTION OF THE SYSTEM (6)

In order to solve the system (6) we first reduce it by means of the substitution

(6') $\xi = x + y + z, \quad \eta = xy + yz + zx, \quad \zeta = xyz$

to the more easily integrable form

$$\xi' + \xi^2 = 3\eta,$$
(7)
$$\eta' = 0,$$
$$\zeta' + 3\xi\zeta = \eta^2.$$

The second of these equations shows that η is constant and we write for later convenience

(8) $$\eta = -\frac{c^2}{3}.$$

Then the first becomes

(8') $$\xi' + \xi^2 + c^2 = 0,$$

the solution of which (unless ξ is constant, which special case is treated later)

(9) $$\xi = c \tan c\,(k - t),$$

where k is an arbitrary constant of integration. (The special case where $c = 0$, that is, $\eta = 0$, is treated separately later.)

By (6') the three functions x, y, z are the roots of the cubic equation

(9') $$x^3 - \xi x^2 + \eta x - \zeta = 0.$$

Substituting

(9'') $$x = X + \frac{1}{3}\xi$$

and using (8) and (9), this reduces to

(10) $$X^3 - \frac{1}{3}c^2 \sec^2 c\,(k - t) \cdot X - \left(\zeta + \frac{2}{27}\xi^3 - \frac{1}{3}\xi\eta\right) = 0.$$

We observe that the third equation in (7) is linear in ζ and has for a particular solution*

(10') $$\zeta = -\frac{2}{27}\xi^3 + \frac{1}{3}\xi\eta,$$

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and hence we obtain its general solution in the form

$$(11) \qquad \zeta = -\frac{2}{27}\,\xi^3 + \frac{1}{3}\,\xi\eta + c_1 c^3 \sec^3 c(k-t),$$

where c_1 is an arbitrary constant of integration. If now we divide the roots of the cubic (10) by $c\sec c(k-t)$, that is, employ the substitution

$$X = a\,c\sec c(k-t),$$

it becomes

$$(12) \qquad a^3 - \frac{1}{3}\,a - c_1 = 0,$$

so that the new unknown a is actually constant and dependent on c_1. We then have as a general solution of equations (6)

$$(13) \qquad x, y, z = \frac{1}{3}\,c\tan c(k-t) + a_i\,c\sec c(k-t),$$

where a_i takes on successively the values which are the roots of the cubic (12). The result thus involves three arbitrary constants c, k, c_1.

THE PRINCIPAL QUADRATIC FORM IN ONE VARIABLE

Going back now to the original notation by means of (5') and integrating the expressions for x, y, z given in (13), we obtain

$$L_i^* = \frac{1}{2}\log \lambda_i^* = \frac{1}{3}\log\cos(k-x_1^*) + a_i\log\{\sec c(k-x_1^*) + \tan c(k-x_1^*)\}.$$

Therefore

$$\lambda_i^* = \cos^{2/3} c(k-x_1^*)\{\sec c(k-x_1^*) + \tan c(k-x_1^*)\}^{2a_i} \quad (i = 2, 3, 4)$$

are the general values of the coefficients in the differential form (4).

We now return to the general orthogonal form (1) by the transformation

$$dx_1^{*2} = \frac{4\,dx_1^2}{c^2(1+x_1^2)},$$

that is,

$$\cos c(k-x_1^*) = \frac{2\,x_1}{1+x_1^2},$$

which eliminates all the trigonometric functions, giving

$$\lambda_1 = \frac{4}{c^2(1+x_1^2)^2}, \qquad \lambda_i = \left(\frac{2x_1}{1+x_1^2}\right)^{2/3} x_1^{2a_i} \qquad (i = 2, 3, 4).$$

We have thus established our main theorem:

The principal solution, in orthogonal form (1) *where the coefficients are functions of x_1 alone, of Einstein's cosmological equations*

$$(2) \qquad\qquad R_{ik} - \frac{1}{4} g_{ik} R = 0,$$

is given by

$$(14) \quad ds^2 = \frac{4\, dx_1^2}{c^2(1+x_1^2)^2} + \left(\frac{2x_1}{1+x_1^2}\right)^{2/3} \{ x_1^{2a_2}\, dx_2^2 + x_1^{2a_3}\, dx_3^2 + x_1^{2a_4}\, dx_4^2 \},$$

where c is an arbitrary constant and a_2, a_3, a_4 are the roots of a cubic of type (12), *that is, are constants obeying the relations*

$$(14') \qquad\qquad a_2 + a_3 + a_4 = 0,$$
$$a_2 a_3 + a_3 a_4 + a_4 a_2 = -\frac{1}{3}.$$

The potentials in this form are in general intercendental (transcendental) functions, since the exponents are usually irrational. A simple example where the exponents are rational numbers and hence the potentials g_{ik} are algebraic functions is given by

$$a_2 = \frac{1}{3}, \qquad a_3 = \frac{1}{3}, \qquad a_4 = -\frac{2}{3}.$$

(This of course does not mean that the finite four-dimensional spread is necessarily algebraic.)

SPECIAL CASE WHERE THE CONSTANT c VANISHES

In this special case $\eta = 0$, so that from (7) we find

$$\xi = \frac{1}{t - t_0} \quad \text{or} \quad \xi = 0.$$

For the first form we find $\zeta = c_2/(t - t_0)^3$, and finally,

$$x = \frac{\lambda}{t - t_0}, \qquad y = \frac{\mu}{t - t_0}, \qquad z = \frac{\nu}{t - t_0},$$
$$(15) \qquad\qquad \lambda + \mu + \nu = 1, \qquad \lambda\mu + \mu\nu + \nu\lambda = 0.$$

For the second form, $\zeta = \text{constant} = c_3^3$, and

(16) $x = c_3,\qquad y = \omega c_3,\qquad z = \omega^2 c_3,$

where ω is an imaginary cube root of unity.

Thus the quadratic form is either

(15′) $ds^2 = dx_1^2 + x_1^{2\lambda}\,dx_2^2\ + x_1^{2\mu}\,dx_3^2\ + x_1^{2\nu}\,dx_4^2,$

or else

(16′) $ds^2 = dx_1^2 + e^{2c_3x_1}\,dx_2^2 + e^{2\omega c_3 x_1}\,dx_3^2 + e^{2\omega^2 c_3 x_1}\,dx_4^2.$

Both of these solutions may be included by transformation in the single form*

(17)

$$ds^2 = x_1^{2a_1}\,dx_1^2 + x_1^{2a_2}\,dx_2^2 + x_1^{2a_3}\,dx_3^2 + x_1^{2a_4}\,dx_4^2,$$

$$a_2 + a_3 + a_4 = a_1 + 1,\qquad a_2^2 + a_3^2 + a_4^2 = (a_1 + 1)^2.$$

The simplest solution in integers of the two conditions on the exponents in (17) is $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, $a_4 = -1$, giving

$$ds^2 = x_1^4(dx_1^2 + dx_2^2 + dx_3^2) + x_1^{-2}\,dx_4^2.$$

By a misprint in the American Journal of Mathematics, the last term is incorrectly written $x_4^{-2}\,dx_4^2$.

SPECIAL CASE WHERE THE FUNCTION ξ IS CONSTANT

Going back to (8′), it was noted that the solution (9) does not include the case where ξ is constant. In that case, replacing ic by k, we have from (7)

(18) $\xi = k,\qquad \eta = \dfrac{k^2}{3},\qquad \zeta = \dfrac{k^3}{27} + C_1 e^{-3kt},$

where C_1 is a new constant of integration. We then find

$$x = \frac{k}{3} + C_1 e^{-kt},$$

(19)

$$y = \frac{k}{3} + \omega C_1 e^{-kt},$$

$$z = \frac{k}{3} + \omega^2 C_1 e^{-kt},$$

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* This result is the solution of the Einstein field equations in their first form, $R_{ik} = 0$, and was first given by the writer in a paper read before the Society in September, 1920 (see Bulletin of the American Mathematical Society, vol. 27 (1920–21), p. 62). See also the American Journal of Mathematics, vol. 43 (1921), p. 220. The same problem was later treated by Murnaghan and Eisenhart. The general results of the present paper were given in Science, vol. 54 (1921), pp. 304–305.

where ω is, as above, an imaginary cube root of unity. From these values, using (5′), we find the three functions L_i^*; and finally by a simple transformation and a change of constants the final form is obtained

$$(20) \qquad ds^2 = \frac{dx_1^2}{k^2 \, x_1^2} + x_1^{2/3} \left\{ e^{C x_1^{-1}} \, dx_2^2 + e^{\omega C x_1^{-1}} \, dx_3^2 + e^{\omega^2 C x_1^{-1}} \, dx_4^2 \right\}.$$

Summary of solutions

For the symmetric system of differential equations (6) *all solutions are included in* (13), (19), (15) *and* (16). We observe that (13) involves three arbitrary constants, (19) and (15) involve two, and (16) involves only one.

For our original problem, all solutions of the cosmological equations are included in (14), (20), *and* (17).

We note that the results involve only elementary functions. In general the potentials, that is the coefficients λ_i in the expression ds^2, are transcendental, but for an infinitude of special values of the constants involved they reduce to algebraic functions.

A form which is equivalent to the main solution (14) is

$$ds^2 = dx_1^2 + \lambda_2 \, dx_2^2 + \lambda_3 \, dx_3^2 + \lambda_4 \, dx_4^2,$$

$$\lambda_i = \left(\frac{1 - T^2}{1 + T^2} \right) \left(\frac{1 - T}{1 + T} \right)^{2\alpha_i}, \qquad T = \tan \frac{c x_1}{2},$$

where α_2, α_3, α_4 are the roots of a cubic equation of the form

$$\alpha^3 - \frac{1}{3} \alpha - c_1 = 0,$$

the constant c_1 being arbitrary as well as c.

Representation in space of seven dimensions

We now prove the following result:

All the solutions obtained may be represented in finite form by manifolds of four dimensions imbedded in a flat space of seven dimensions, the finite equations involving integrals of elementary functions.

We first observe that all quadratic forms of the type

$$(21) \qquad ds^2 = \lambda_1(x_1) \, dx_1^2 + \lambda_2(x_1) \, dx_2^2 + \lambda_3(x_1) \, dx_3^2 + \lambda_4(x_1) \, dx_4^2$$

are of class three, that is, are reducible to the sum of seven squares of exact differentials.

We may write (21) as follows:

(21') $$ds^2 = ds_2^2 + ds_3^2 + ds_4^2 = Q_2 + Q_3 + Q_4,$$

where

$$ds_i^2 = Q_i = \frac{1}{3}\,\lambda_1(x_1)\,dx_1^2 + \lambda_i(x_1)\,dx_i^2 \qquad (i = 2, 3, 4).$$

Here Q_i is a quadratic form in only two variables, x_1 and x_i, and is the fundamental form of a surface of revolution. *Thus our element* (21) *is the sum of the elements of three surfaces of revolution.*

Since the coördinate x_1 is common to the three forms Q_i we may take the three surfaces about a common axis of revolution. Denote this by X_1, and construct the first surface in a three-flat $X_1\,X_2\,X_3$, the second surface in a three-flat $X_1\,X_4\,X_5$, the third surface in a three-flat $X_1\,X_6\,X_7$. Here $X_1, X_2, \cdots, X_7$ are cartesian coördinates in a flat space of seven dimensions. The three surfaces of rotation thus have a common axis but are contained in three distinct three-flats which are mutually orthogonal.

The general theory of what I have called separable or summable* quadratic differential forms in any number of variables is a subject of considerable interest. The problem is to represent the given form in n variables as the sum of two or more forms each involving fewer than n variables. We must distinguish the case where the variables used in the summands are completely independent, and the case where dependent variables are introduced. The problem discussed in the present paper comes under the latter case (since the three binary forms in (21') do not involve completely independent variables), while the problem of an earlier paper in these Transactions† gives an example of complete separability.

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* Kasner, *Separable quadratic differential forms and Einstein solutions*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 95–96, and abstracts in the Bulletin of the American Mathematical Society.

† *An algebraic solution of the Einstein equations,* vol. 27 (1925), pp. 101–105.

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